COMBINATORICA
Akadémiai Kiadó – Springer-Verlag

COMBINATORICA 16 (3) (1996) 331–341

BOUNDING THE NUMBER OF CIRCUITS OF A GRAPH

GUOLI DING

Received July 7, 1994

Let $c(G)$ denote the number of circuits of a graph G. In this paper, we characterize those minor-closed classes $\mathcal{G}$ of graphs for which there is a polynomial function $p(.)$ such that $c(G) \le p(|E(G)|)$ for all graphs G in $\mathcal{G}$.

1. Introduction

Perhaps the most simple way to solve an optimization problem is to list all its feasible solutions and then choose among them the best. In general, this exhaustive algorithm is not very efficient simply because there are too many feasible solutions. However, under certain circumstances, it solves the problem very quickly.

Suppose one is interested in finding the best circuit, with respect to certain criteria, of a graph G. Depending on the criteria, one may design different kinds of algorithms to solve this problem. However, no matter what the criteria are, if $c(G)$, the number of circuits of G, is relatively small, one may always solve the problem by using the exhaustive algorithm. Clearly, the exhaustive algorithm runs in polynomial time if and only if $c(G)$ is bounded by a polynomial function of $|E(G)|$ (of course we need to assume that it takes polynomial time to compare any two circuits, with respect to the given criteria). Yet we observe that for a fixed graph G, it does not make any sense to say $c(G)$ is relatively big or relatively small because both $c(G)$ and $|E(G)|$ are fixed numbers. Therefore, we need to consider classes $\mathcal{G}$ of graphs instead of individual graphs. The main question to be answered in this paper is: for which $\mathcal{G}$, do there exist a polynomial function $p(.)$ depending only on $\mathcal{G}$ such that $c(G) \le p(|E(G)|)$ for all graphs G in $\mathcal{G}$? If such a polynomial exists, we will call $\mathcal{G}$ a *poly-class*.

Mathematics Subject Classification (1991): 05 B 35, 05 C 55, 05 C 10

Let C_n and L_n be graphs as illustrated in Figure 1. Let $\mathscr{C} = \{C_n : n \geq 3\}$ and let $\mathscr{L} = \{L_n : n \geq 2\}$. The graphs in $\mathscr{C}$ and $\mathscr{L}$ shall be called *double circuits* and *cross ladders*, respectively. It is not difficult to check that $c(C_n) = 2^n + n$ and $c(L_n) = 2^{n+1} - 3n - 1$. Thus a necessary condition for a class of graphs to be a poly-class is that it contains only finitely many graphs in $\mathscr{C} \cup \mathscr{L}$.

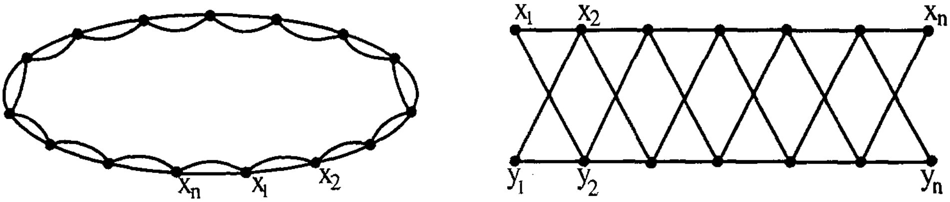

Figure 1. A double circuit C_n and a cross ladder L_n

A graph H is a *minor* of a graph G if H can be obtained from a subgraph of G by contracting edges. Clearly, $c(H) \leq c(G)$ if H is a minor of G. To simplify our question, we assume that $\mathscr{G}$ is *minor-closed*. That is, a minor of a member of $\mathscr{G}$ is also a member of $\mathscr{G}$. Our main result of this paper says that if we only consider minor-closed classes $\mathscr{G}$ of graphs, then the necessary condition mentioned above is also sufficient for $\mathscr{G}$ to be a poly-class.

(1.1) Theorem. *Let $\mathscr{G}$ be a minor-closed class of graphs. Then $\mathscr{G}$ is a poly-class if and only if $\mathscr{G} \cap (\mathscr{C} \cup \mathscr{L})$ is finite.*

In fact, (1.1) is not the first theorem of this kind. Let M_n be a matching with n edges and let $\mathscr{M} = \{M_n : n \geq 1\}$. Let $i(G)$ denote the number of maximal independent vertex-sets of a simple graph G. Clearly, $i(M_n) = 2^n$ for all $n \geq 1$. The next is a reformulation of a result due to Balas and Yu [1].

(1.2) *Let $\mathscr{G}$ be a class of simple graphs that is closed under taking induced subgraphs. Then there is a polynomial function $p(.)$ such that $i(G) \leq p(|V(G)|)$ for all graphs G in $\mathscr{G}$ if and only if $\mathscr{G} \cap \mathscr{M}$ is finite.*

There are a few more results of this kind. Let $t(G)$ denote the number of spanning trees of a graph G. For $i = 1,2,3,4,5$, let each T_n^i be the graph as illustrated in Figure 2. Clearly, $t(K_{2,n}) = n2^{n-1}$ and $t(T_n^i) = 2^n$ for all i. Let $\mathscr{T}$ be the union of $\{K_{2,n} : n \geq 1\}$ and $\{T_n^i : i = 1,2,3,4,5, \text{ and } n \geq 3\}$. Then the following holds [3].

(1.3) *Let $\mathscr{G}$ be a class of graphs that is closed under taking topological minors. Then there is a polynomial function $p(.)$ such that $t(G) \leq p(|E(G)|)$ for all graphs G in $\mathscr{G}$ if and only if $\mathscr{G} \cap \mathscr{T}$ is finite.*

If $\mathscr{G}$ in (1.3) is minor-closed, then we can make a stronger statement. Let $b(M)$ denote the number of bases of a matroid M. Let $U_{1,2}$ be the two-element uniform

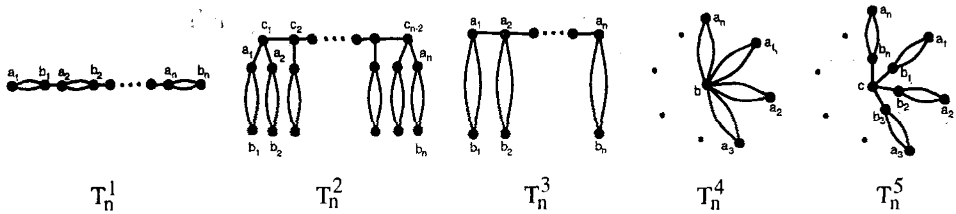

Figure 2. These graphs have many spanning trees

matroid of rank one. Let U_n be the direct sum of n mutually disjoint copies of $U_{1,2}$. Obviously, $b(U_n)=2^n$ for all $n\geq 1$. Let $\mathcal{U}=\{U_n : n\geq 1\}$. The following is another result of the author [4].

(1.4) *Let F be a fixed finite field and let $\mathcal{F}$ be a class of F-representable matroids that is closed under taking minors. Then there is a polynomial function $p(.)$ such that $b(M)\leq p(|E(M)|)$ for all matroids M in $\mathcal{F}$ if and only if $\mathcal{F}\cap\mathcal{U}$ is finite.*

Let P_n denote the *double path* of length n, the graph as illustrated in Figure 3. Then let $\mathcal{P}=\{P_n : n\geq 0\}$ and let $a(G)$ be the number of paths of a graph G. It is not difficult to check that $a(P_n)=2^{n+2}-n-3$ for all $n\geq 0$. The author proved the following in [5].

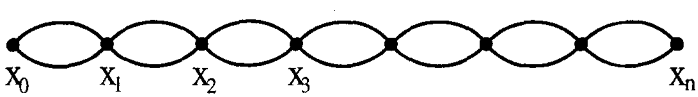

Figure 3. A double path P_n

(1.5) *Let $\mathcal{G}$ be a minor-closed class of graphs. Then there is a polynomial function $p(.)$ such that $a(G)\leq p(|V(G)|+|E(G)|)$ for all graphs G in $\mathcal{G}$ if and only if $\mathcal{G}\cap\mathcal{P}$ is finite.*

From the algorithmic point of view, theorems presented in this section can be considered as characterizations of classes of various combinatorial objects for which the exhaustive algorithm runs in polynomial time. The rest of the paper is to prove (1.1). We shall prove, essentially, that a "well connected" graph with a long double path minor must have C_n or L_n as a minor for some large n.

2. Producing a long double circuit minor

In this section, we derive a sufficient condition under which the existence of a long double path minor implies the existence of a long double circuit minor. Naturally, only loopless graphs are considered in this section.

Let G be a graph. For any $F \subseteq E(G)$ and $Z \subseteq V(G)$, the graphs obtained from G by deleting edges in F, contracting edges in F, and deleting vertices in Z shall be denoted by $G\backslash F$, G/F, and $G-Z$, respectively. If $F=\{e\}$ or $Z=\{z\}$, then we denote $G\backslash F$, G/F, and $G-Z$ by $G\backslash e$, G/e, and $G-z$, respectively. Besides, we will denote by $[n]$ the set $\{1,\ldots,n\}$ while $[0]$ will be interpreted as the empty set.

Let G be a 2-connected graph. A *2-separation* of G is an ordered pair (G_1,G_2) of subgraphs of G for which $E(G_1)=E(G)-E(G_2)$, $|V(G_1)\cap V(G_2)|=2$, and both $V(G_1)$ and $V(G_2)$ are proper subsets of $V(G)$. Clearly, (G_2,G_1) is also a 2-separation of G if (G_1,G_2) is. Notice that the last requirement in our definition makes our 2-separations differ from ordinary ones. A 2-separation (G_1,G_2) *separates* two vertices x and y of G if

$$|\{x,y\}\cap(V(G_1)-V(G_2))| = |\{x,y\}\cap(V(G_2)-V(G_1))| = 1.$$

A sequence $(G_{1,1},G_{1,2}),\ldots,(G_{m,1},G_{m,2})$ of 2-separations of G is *nested* if either the sequence does not have any term or m is positive such that for every i in $[m-1]$, $G_{i,1}$ is a subgraph of $G_{i+1,1}$ and $V(G_{i,1})\cap V(G_{i,2}) \neq V(G_{i+1,1})\cap V(G_{i+1,2})$. Finally, a P_n minor of G is *between* x and y if x and y are contracted to the two end-vertices of P_n. Now the main result of this section can be stated as follows.

(2.1) Theorem. *Let m and n be integers for which $m \geq 0$ and $n \geq 2$. Let x and y be two vertices of a 2-connected graph G. Suppose that G has a $P_{3n^3(m+1)}$ minor between x and y. In addition, suppose that every nested sequence of 2-separations separating x and y has at most m terms. Then G has C_{n+1} as a minor.*

We shall break the proof of (2.1) into the proofs of a few lemmas. We first recall a theorem of Dilworth [2].

(2.2) *A poset can be partitioned into k chains if k is the size of its largest antichain.*

The following is an immediate corollary of (2.2).

(2.3) *Let n be a nonnegative integer. Then a poset with n^2 elements must have an antichain of size $n+2$ or a chain of size n.*

An *$\{x,y\}$-path* is a path for which the two end-vertices are x and y. If X and Y are sets of vertices, then a path is *between* X and Y if it is an $\{x,y\}$-path for some x in X and y in Y. The following is a corollary of (2.3).

(2.4) *Let P^1, P^2, and P^3 be internally vertex-disjoint $\{x,y\}$-paths of a graph G. Suppose for some integer n exceeding one, that $G-V(P^3)$ has n^2 vertex-disjoint paths between $V(P^1)-\{x,y\}$ and $V(P^2)-\{x,y\}$. Then G has C_{n+1} as a minor.*

Proof. Let $x,x_1,\ldots,x_r,y$ be the vertices of P^1, and let $x,y_1,\ldots,y_s,y$ be the vertices of P^2, where the vertices are listed in the order as in the paths. Let $Q_1,\ldots,Q_{n^2}$ be vertex-disjoint paths of $G-V(P^3)$ between $V(P^1)-\{x,y\}$ and $V(P^2)-\{x,y\}$. Then each Q_i is an $\{x_{r_i},y_{s_i}\}$-path for some $r_i \in [r]$ and $s_i \in [s]$.

Let $\mathcal{Q}=\{Q_1,\ldots,Q_{n^2}\}$. Then for any two paths Q_i and Q_j in $\mathcal{Q}$, let $Q_i \preceq Q_j$ if $r_i \leq r_j$ and $s_i \leq s_j$. It is clear that $\mathcal{Q}$ is partially ordered by $\preceq$. By (2.3), there is a subset $\mathcal{Q}'$ of $\mathcal{Q}$ such that $\mathcal{Q}'$ is either an antichain of size $n+2$ or a chain of size n. In both cases, it is not difficult to see that the subgraph of G that consists of P^1, P^2, P^3, and the members of $\mathcal{Q}'$ has C_{n+1} as a minor. ∎

We also need Menger's Theorem in the following form. Suppose that G_1 and G_2 are two subgraphs of a graph G for which

$$V(G_1) \cap V(G_2) = \emptyset, \quad V(G_1) \cup V(G_2) = V(G), \quad \text{and} \quad E(G_1) \cup E(G_2) = E(G).$$

Then (G_1, G_2) is called a 0-*separation* of G. If X and Y are subsets of $V(G_1)$ and $V(G_2)$, respectively, then we say that (G_1, G_2) *separates* X and Y.

(2.5) *Let X and Y be disjoint sets of vertices of a graph G. Suppose that G has at most k vertex-disjoint paths between X and Y. Then there exists a subset S of $V(G)$ of size at most k such that $G-S$ has a 0-separation separating $X-S$ and $Y-S$.*

With the preparations above, now we can prove the following lemma.

(2.6) *Let n be an integer exceeding one. If G is a graph for which there are two vertices x and y such that G has three internally vertex-disjoint $\{x,y\}$-paths and a P_{3n^3} minor between x and y, then G has C_{n+1} as a minor.*

Proof. Let P^1, P^2, and P^3 be three internally vertex-disjoint $\{x,y\}$-paths of G. For each $i=1,2,3$, let $X_i = V(P^i)-\{x,y\}$. By (2.4), we may assume that $G-\{x,y\}$ has at most $2n^2-2$ vertex-disjoint paths between X_1 and $X_2 \cup X_3$. Thus, by (2.5), there exists a subset S_1 of $V(G)$ such that $\{x,y\} \subseteq S_1$, $|S_1| \leq 2n^2$, and $G-S_1$ has a 0-separation (G_1, G') separating X_1-S_1 and $(X_2 \cup X_3)-S_1$. Without loss of generality, let us assume that $X_1 - S_1 \subseteq V(G_1)$. Now by (2.4), we may assume that G' has at most n^2-1 vertex-disjoint paths between X_2-S_1 and X_3-S_1. Therefore, by (2.5) again, we deduce that there exists a subset S_2 of $V(G')$ such that $|S_2| \leq n^2-1$ and $G'-S_2$ has a 0-separation (G_2, G_3) separating X_2-S and X_3-S, where $S=S_1 \cup S_2$. Without loss of generality, let us assume that $X_2 - S \subseteq V(G_2)$ and $X_3 - S \subseteq V(G_3)$. Clearly, $|S| \leq 3n^2-1$ and $G-S$ is the disjoint union of G_1, G_2, and G_3.

Since G has a P_{3n^3} minor between x and y, there exist vertex-disjoint connected subgraphs $H_0, H_1, \ldots, H_{3n^3}$ of G such that $x \in V(H_0)$, $y \in V(H_{3n^3})$, and for all i in $[3n^3]$, there are two edges e_i', e_i'' between $V(H_{i-1})$ and $V(H_i)$. Let I be the set of indices i for which $V(H_i) \cap S \neq \emptyset$. From the choice of S it is clear that both 0 and $3n^3$ are in I. Let $J=[3n^3]-I$ and let $j \in J$. Then H_j is a subgraph of $G-S$. Since it is connected, H_j is a subgraph of some G_k. Moreover, if $j-1$ is also in J, then e_j' and e_j'' must both belong to $E(G_k)$. This implies that H_{j-1} is also a subgraph of G_k. Therefore, whenever i_1 and i_2 are members of I such that $i_2 - i_1 > 1$ and

$[i_2-1]-[i_1] \subseteq J$, there must be a G_k containing H_j as a subgraph for all j in $[i_2-1]-[i_1]$.

Since 0 and $3n^3$ are in I, the set J can be partitioned into fewer than $|I|$ consecutive sets of integers. Notice that $|I| \leq |S| \leq 3n^2-1$. Thus $|J| > 3n^3 - |I| > (n-1)(|I|-1)$. It follows that there are two members i_1 and i_2 of I such that $[i_2-1]-[i_1] \subseteq J$ and $i_2-1-i_1 \geq n$. Clearly, there is a graph G_k containing H_j as a subgraph for all j in $[i_2-1]-[i_1]$. Let $k' \in \{1,2,3\}-\{k\}$. Then it is clear that $V(P^{k'})$ does not meet any of $V(H_j)$ for j in $[i_2-1]-[i_1]$. Let $i'_1 \in [i_1] \cup \{0\}$ be maximum and $i'_2 \in [3n^3]-[i_2-1]$ be minimum such that both $V(H_{i'_1})$ and $V(H_{i'_2})$ meet $V(P^{k'})$. Now it is clear that the subgraph of G that consists of $P^{k'}$, subgraphs H_j with $i'_1 \leq j \leq i'_2$, and all edges e'_j, e''_j with $i'_1 < j \leq i'_2$ has C_{n+1} as a minor. ∎

Let (G_1, G_2) be a 2-separation of a 2-connected graph G. Let $V(G_1) \cap V(G_2) = \{u,v\}$. Then, for $i=1,2$, we define G_i^* to be the graph obtained from G_i by adding a new edge between u and v. Furthermore, if H is a subgraph of G, then for each i in $\{1,2\}$, let H^i be the subgraph of G_i^* induced by $V(H) \cap V(G_i)$. The following is an obvious observation.

(2.7) *Let (G_1, G_2) be a 2-separation of a 2-connected graph G. If H is a connected subgraph of G, then H^i is a connected subgraph of G_i^* if $V(H) \cap V(G_i) \neq \emptyset$.*

(2.8) *Let x and y be two vertices of a 2-connected graph G. Suppose that G has a 2-separation (G_1, G_2) separating x and y. Also, for some positive integer k, suppose that G has a P_k minor between x and y. Let $V(G_1) \cap V(G_2) = \{u,v\}$. Then, for some w in $\{u,v\}$, G_1^* has a P_{k_1} minor between x and w, and G_2^* has a P_{k_2} minor between w and y, where k_1 and k_2 are nonnegative integers for which $k_1 + k_2 = k$.*

Proof. Since G has a P_k minor between x and y, there exist vertex-disjoint connected subgraphs $H_0, H_1, \ldots, H_k$ of G such that $x \in V(H_0)$, $y \in V(H_k)$, and for all i in $[k]$, there are two edges e'_i, e''_i between $V(H_{i-1})$ and $V(H_i)$. We first prove the following.

(1) *If $\{e'_i, e''_i\} \cap E(G_1) \neq \emptyset$, than $\{e'_j, e''_j\} \subseteq E(G_1)$ for all j in $[i-1]$.*

Let e_i be a member of $\{e'_i, e''_i\} \cap E(G_1)$ and let e_j be a member of $\{e'_j, e''_j\}$ for some j in $[i-1]$. Since e_i and e_j are used in a double path minor between x and y, G must have an $\{x,y\}$-path P using both e_i and e_j. Let X, Y, and Z be the three connected components of $P \backslash \{e_i, e_j\}$. If e_j is not in $E(G_1)$, then X, Y, and Z constitute a set of three mutually vertex-disjoint paths between $V(G_1)$ and $V(G_2)$. This is certainly impossible since (G_1, G_2) is a 2-separation. Therefore, e_j must be in $E(G_1)$ and thus (1) is proved.

From (1) we conclude that there exists i in $[k]$ such that $\{e'_j, e''_j\} \subseteq E(G_1)$ for all j in $[i-1]$, and $\{e'_j, e''_j\} \subseteq E(G_2)$ for all j in $[k]-[i]$. Now we define k_1 as follows. If $\{e'_i, e''_i\} \cap E(G_j) \neq \emptyset$ for all j in $\{1,2\}$, then let $k_1 = i$. If $\{e'_i, e''_i\} \cap E(G_j) \neq \emptyset$ for only one j in $\{1,2\}$, then let $k_1 = i - j + 1$. It is straightforward to verify that

(2) *$\{e'_{k_1}, e''_{k_1}\} \cap E(G_1) \neq \emptyset$ if $k_1 \neq 0$, and $\{e'_{k_1+1}, e''_{k_1+1}\} \cap E(G_2) \neq \emptyset$ if $k_1 \neq k$.*

Consequently, $V(H_{k_1})$ meets both $V(G_1)$ and $V(G_2)$. Therefore, $V(H_{k_1}) \cap \{u,v\} \neq \emptyset$. Let $w \in V(H_{k_1}) \cap \{u,v\}$. Now we prove that G_1^* has a P_{k_1} minor between x and w. The proof of "G_2^* has a P_{k-k_1} minor between w and y" is similar.

Without loss of generality, let us assume that $k_1 > 0$ and $e'_{k_1} \in E(G_1)$. Then for each integer j with $0 \leq j \leq k_1$, we must have $V(H_j) \cap V(G_1) \neq \emptyset$. This claim is clear if $j = 0$ because x is in $V(H_0) \cap V(G_1)$. If j is in $[k_1]$, then by (1) and our assumption that $e'_{k_1} \in E(G_1)$, the edge e'_j must be in $E(G_1)$. Therefore, the vertices of H_j that are incident with e'_j are in $V(G_1)$, as required. Now by (2.7), $H_0^1, \ldots, H_{k_1}^1$ are connected subgraphs of G_1^*. Since $V(H_j^1) = V(H_j) \cap V(G_1)$ for all $j = 0, 1, \ldots, k_1$, it follows that

(3) *e'_j and e''_j are between $V(H_{j-1}^1)$ and $V(H_j^1)$ for all j in $[k_1 - 1]$, and e'_{k_1} is between $V(H_{k_1-1}^1)$ and $V(H_{k_1}^1)$.*

If e''_{k_1} is in $E(G_1)$, then it is clear that the subgraph of G_1^* consisting of $H_0^1, \ldots, H_{k_1}^1$ and $e'_1, e''_1, \ldots, e'_{k_1}, e''_{k_1}$ has a P_{k_1} minor between x and w. If e''_{k_1} is in $E(G_2)$, then both $V(H_{k_1-1})$ and $V(H_{k_1})$ meet $\{u,v\}$. Therefore, the new edge e of G_1^* is between $V(H_{k_1-1}^1)$ and $V(H_{k_1}^1)$. Now it is clear that the subgraph of G_1^* consisting of $H_0^1, \ldots, H_{k_1}^1$ and $e'_1, e''_1, \ldots, e'_{k_1-1}, e''_{k_1-1}, e'_{k_1}, e$ has a P_{k_1} minor between x and w. The proof of (2.8) is complete. ∎

Finally, we are ready to prove (2.1).

Proof of (2.1). We proceed by induction on m. If $m = 0$, then there is no 2-separation separating x and y. It follows that G has three internally vertex-disjoint $\{x,y\}$-paths. Thus the result follows from (2.6).

Suppose now there is at least one 2-separation separating x and y. Choose such a 2-separation (G_1, G_2) for which $|V(G_1)|$ is minimum. Let $V(G_1) \cap V(G_2) = \{u,v\}$. Then, by (2.8), there is a vertex w in $\{u,v\}$ such that G_1^* has a P_{k_1} minor between x and w, and G_2^* has a P_{k_2} minor between w and y, where k_1 and k_2 are nonnegative integers for which $k_1 + k_2 = 3n^3(m+1)$. Clearly, we have

(1) *G_1^* and G_2^* are 2-connected, and*

(2) *G_1^* and G_2^* are minors of G.*

From the minimality of $|V(G_1)|$, we also observe that

(3) *there is no 2-separation of G_1^* separating x and w.*

Furthermore, whenever $(G_{1,1},G_{1,2})$, ..., $(G_{k,1},G_{k,2})$ is a nested sequence of 2-separations of G_2^* separating w and y, (G_1,G_2), $(G_{1,1},G_{1,2})$, ..., $(G_{k,1},G_{k,2})$ must be a nested sequence of 2-separations of G separating x and y. Thus we conclude that

(4) *every nested sequence of 2-separations of G_2^* separating w and y has at most $m-1$ terms.*

Since $k_1+k_2=3n^3(m+1)$, we must have $k_1\geq 3n^3$ or $k_2\geq 3n^3m$. But in either case, we deduce from (1), (3), (4), and our induction hypothesis that G_1^* or G_2^* has C_{n+1} as a minor. Thus, by (2), G has C_{n+1} as a minor. The proof of (2.1) is complete. ∎

3. A proof of Theorem (1.1)

Let e_1 and e_2 be two edges of a 2-connected graph G. Then a 2-separation (G_1,G_2) of G *separates* e_1 and e_2 if

$$|\{e_1,e_2\}\cap E(G_1)| = |\{e_1,e_2\}\cap E(G_2)| = 1,$$

and each e_i is incident with at most one of the two vertices in $V(G_1)\cap V(G_2)$. Let Δ be the graph obtained from K_3 by adding a parallel edge. If G has Δ or K_4 as a minor such that e_1 and e_2 appear in the way as illustrated in Figure 4, then we say that G has $\Delta(e_1,e_2)$ or $K_4(e_1,e_2)$, respectively, as a minor. We begin by proving the following lemma.

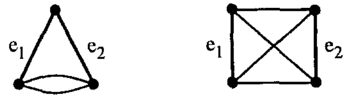

Figure 4. $\Delta(e_1,e_2)$ and $K_4(e_1,e_2)$

(3.1) *Let e_1 and e_2 be two non-parallel edges of a 2-connected graph G such that there are no 2-separations separate them. Suppose that $G\setminus e$ is 2-connected for every edge $e\in E(G)-\{e_1,e_2\}$. Then G has $\Delta(e_1,e_2)$ or $K_4(e_1,e_2)$ as a minor.*

Proof. Since G is 2-connected, there must be a circuit C of G containing both e_1 and e_2. Let X and Y be the two paths obtained from C by deleting e_1 and

e_2. Let $x_1,\ldots,x_m$ be all vertices of X and let $y_1,\ldots,y_n$ be all vertices of Y, where the vertices are listed in the order as they appear in the paths. Without loss of generality, let us assume that $e_1=x_1y_1$ and $e_2=x_my_n$.

A subgraph H of G is a *bridge* (with respect to C) if either H consists of a single edge of $E(G)-E(C)$ for which the end-vertices are in $V(C)$, or H is obtained from a connected component J of $G-V(C)$ by adding all edges between $V(C)$ and $V(J)$. Vertices in $V(C)\cap V(H)$ shall be called the *attachments* of H. Since G is 2-connected, G has no bridge with fewer than two attachments. Suppose G has a bridge H for which there are two attachments u and v such that $\{u,v\}\subseteq V(X)$ or $\{u,v\}\subseteq V(Y)$. Let P be a $\{u,v\}$-path of H. Then it is easy to see that the subgraph of G consisting of C and P has $\Delta(e_1,e_2)$ as a minor. Thus, we may assume that

(1) *every bridge has exactly two attachments, one is in $V(X)$ and one is in $V(Y)$.*

A bridge H is called an *(i,j)-bridge* if x_i and y_j are the two attachments of H. Since e_1 and e_2 are not in parallel, C must have an edge e other than e_1 and e_2. By the assumption of (3.1), $G\backslash e$ is 2-connected. It follows that G has an (i,j)-bridge H for which $(i,j)\neq(1,1)$ or (m,n). Let $\mathscr{B}_1$ be the set of (i',j')-bridges for which $i'\leq i$ and $j'\leq j$. Also, let $\mathscr{B}_2$ be the set of (i',j')-bridges for which $i'\geq i$, $j'\geq j$, and $i'+j'>i+j$. Now we show that there is a bridge not in $\mathscr{B}_1\cup\mathscr{B}_2$. Let $k\in\{1,2\}$. Then let C^k be the $\{x_i,y_j\}$-path of C for which $e_k\in E(C^k)$. Also, let G_k be the subgraph of G that consists of C^k and the bridges in $\mathscr{B}_k$. Clearly, (G_1,G_2) would be a 2-separation of G that separates e_1 and e_2 if there are no other bridges. Therefore, there is an (i',j')-bridge H' such that $(i'-i)(j'-j)<0$. Let P be an $\{x_i,y_j\}$-path of H and let P' be an $\{x_{i'},y_{j'}\}$-path of H'. Then it is easy to see that the subgraph of G that consists of C, P, and P' has $K_4(e_1,e_2)$ as a minor. The proof of (3.1) is complete. ∎

The next is another lemma we need in proving our main result.

(3.2) *Let G be a 2-connected graph such that $G\backslash e$ is still 2-connected for every edge e of G. Suppose that G has a nested sequence of 2-separations with at least $2n-1$ terms, where n is an integer exceeding one. Then G has C_{n+1} or L_{n+1} as a minor.*

Proof. Choose a nested sequence $(G_{1,1},G_{1,2})$, ..., $(G_{m,1},G_{m,2})$ of 2-separations of G such that m is maximum. Intuitively, these 2-separations separate G into $m+1$ graphs, each of which is a minor of G. We shall make it more precise as follows. Let $H_1=G^*_{1,1}$ and let $H_{m+1}=G^*_{m,2}$. Then, for each i in $[m-1]$, let $V_i=V(G_{i,1})\cap V(G_{i,2})$ and let H_{i+1} be the graph obtained from $(G^*_{i+1,1}\backslash E(G_{i,1}))-(V(G_{i,1})-V_i)$ by adding an edge between the two vertices u_i and v_i of V_i. Furthermore, for each H_i, we choose two edges from it as follows. Suppose $i=1$. First, let $e_{1,1}$ be an edge of H_1 that is between u_1 and v_1. Since $V(H_1)\neq V_1$, the graph H_1 must have an edge not parallel to $e_{1,1}$. Let $e_{1,2}$ be such an edge. If $i=m+1$, then $e_{i,1}$ and $e_{i,2}$ are chosen similarly. If i is between 1 and $m+1$, let $e_{i,1}$ be an edge of H_i between u_{i-1} and v_{i-1}, and let $e_{i,2}$ be an edge of H_i between u_i and v_i.

Let us consider the graph H_i. Since G is 2-connected, it is clear that H_i is also 2-connected. From the definition of a nested sequence of 2-separations and the choice of each $e_{i,j}$ we conclude that $e_{i,1}$ and $e_{i,2}$ are not parallel. In addition, we deduce from the maximality of m that H_i does not have a 2-separation separating $e_{i,1}$ and $e_{i,2}$. Since $G\setminus e$ is 2-connected for every edge e of G, it follows that $H_i\setminus e$ is also 2-connected for every edged e in $E(H_i)-\{e_{i,1},e_{i,2}\}$. Therefore, we conclude from (3.1) that H_i has $\Delta(e_{i,1},e_{i,2})$ or $K_4(e_{i,1},e_{i,2})$ as a minor.

Let m_1 be the number of indices i for which H_i has $\Delta(e_{i,1},e_{i,2})$ as a minor. If $m_1\geq n+1$, then it is clear that G has C_{n+1} as a minor. Thus we may assume that $m_1\leq n$. Since $m\geq 2n-1$, it follows that there are at least n indices i for which H_i has $K_4(e_{i,1},e_{i,2})$ as a minor. In this case, clearly, G has L_{n+1} as a minor. ∎

We also need the following immediate corollary of (1.6) and (4.3) of [5].

(3.3) *For every positive integer k there exist positive integers α_k and β_k having the following property. If a graph G does not have P_k as a minor, then $a(G)\leq|V(G)|+\alpha_k|E(G)|^{\beta_k}$.*

Proof of (1.1). The "only if" part is clear and so we only need to consider the "if" part. Let n be an integer exceeding one such that C_{n+1} and L_{n+1} are not in $\mathscr{G}$. Let $\alpha=\alpha_{3n^3(2n-1)}$ and $\beta=\beta_{3n^3(2n-1)}$. Let $p(x)=\alpha x^{\beta+1}$. We shall prove that $c(G)\leq p(|E(G)|)$ for all graphs G in $\mathscr{G}$.

We proceed by induction on $|V(G)|+|E(G)|$. The result is clear if $|V(G)|+|E(G)|=0$ and hence we shall assume that $|V(G)|+|E(G)|>0$. If G is not 2-connected, let $G_1,\ldots,G_t$ be its maximal 2-connected subgraphs. Clearly, $c(G)=c(G_1)+\ldots+c(G_t)$ and $(E(G_1),\ldots,E(G_t))$ is a partition of $E(G)$. Since $\beta\geq 1$, we conclude from our induction hypothesis that

$$c(G)\leq \alpha|E(G_1)|^{\beta+1}+\ldots+\alpha|E(G_t)|^{\beta+1}\leq p(|E(G)|),$$

as required.

Now let us assume that G is 2-connected. If G has an edge e for which $G\setminus e$ is not 2-connected, then it is not difficult to see that $c(G)=c(G/e)$. Again, the result follows from our induction hypothesis. Thus we may assume that for every edge e of G, the graph $G\setminus e$ is still 2-connected.

Since C_{n+1} and L_{n+1} are not in $\mathscr{G}$ and $\mathscr{G}$ is minor-closed, G does not have C_{n+1} or L_{n+1} as a minor. It follows from (3.2) that no nested sequence of 2-separations of G can have $2n-1$ terms. Thus we deduce from (2.1) that G does not have $P_{3n^3(2n-1)}$ as a minor. From (3.3) and our choice of α and β we know that $a(G)\leq|V(G)|+\alpha|E(G)|^{\beta}$. But every vertex of G is a path of G, so G has at most $\alpha|E(G)|^{\beta}$ paths P for which $E(P)\neq\emptyset$.

For each circuit C of G, let us fix an edge e_C of C. Clearly, since α and β are positive integers, we may assume that G is not a loop. Suppose that each set of parallel edges of G has at most k members. Then for every path P with $E(P) \neq \emptyset$, it is clear that G has at most k circuits C for which $P = C \setminus e_C$. Therefore, $c(G) \leq k\alpha|E(G)|^\beta \leq p(|E(G)|)$, as we wanted. ∎

Acknowledgement. The author thanks Paul Seymour for many helpful discussions. He also thanks National Science Foundation for its support under Grant DMS-9400946.

References

[1] E. BALAS, and C.S. YU: On graphs with polynomially solvable maximum-weight clique problem, *Networks*, **19** (1989), 247–253.

[2] R. P. DILWORTH: A decomposition theorem for partially ordered sets, *Ann. Math.* **51** (1950), 161–166.

[3] G. DING: Graphs with not too many spanning trees, *Networks*, **25** (1995), 193–197.

[4] G. DING: Bounding the number of bases of a matroid, *Combinatorica*, **15** (2) (1995), 159–165.

[5] G. DING: Excluding a long double path minor, *J. Combinatorial Theory* (B), **66** (1) (1996), 11–23.

Guoli Ding

Department of Mathematics,
Louisiana State University,
Baton Rouge, LA 70803-4918
ding@math.lsu.edu